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Fixed Point on Controlled Strong b-Multiplicative Metric Space

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Abstract: In [21] Dania Santina introduced a generalization of a metric space called controlled strong b-metric space (CSbMS) and proved a fixed point result in CSbMS. In [22], Tayyab Kamran explained the notion of b-multiplicative metric space and proved some fixed point theorems in b-multiplicative metric space. The purpose of this paper is to establish the concept of controlled strong b-multiplicative metric space (CSbMMS) and to prove a fixed point result on CSbMMS.

Keywords: *b-Metric Space, Strong Metric Space, Multiplicative Metric Space, Controlled Strong B-Metric Space, Controlled Strong b-Multiplicative Metric Space. 2020 AMS Subject Classification: 47H10, 54H25*

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I. INTRODUCTION

The Banach contraction principle is the foundation of fixed point theory which declares that all contractive mappings on metric spaces always have some fixed points, that are unique. In this sequence many researchers generalized the concept of Banach and established different kind of contractions [1–5]. In this paper, we emphasize on a generalized metric structure called b-metric space (bMS) and its subsequent generalization. Bakhtin [6] and Czerwik [7] step ahead in the generalization of (MS) by introducing the notion b-metric space (bMS). In which altering the triangle inequality of a (MS) has been taken place. Infact, every metric space(MS) is a b-metric space (bMS) when the constant b of the triangle inequality is one. While the reverse of it is not implied. After that, many authors found many fixed-point results in b-metric spaces; [8–20]. As far as this paper is concern, our object in the current study, we refer to papers that contain some generalizations of b-metric spaces, such as a strong b-metric space (SbMS) ([25]). Subsequently, we generalized the concept of controlled strong b-metric space (CSbMS) and put the idea of controlled strong b-multiplicative metric space (CSbMMS) In the main results section, we prove the existence of the fixed point of self-mapping on a complete (CSbMMS) and its uniqueness.

II. PRELIMANARIES

➢ *Definition 2.1 [22]*

Let A be a set having at least one element and take a real no. $s \geq 1$. A function $\mu: A \times A \rightarrow [1, \infty)$ is said to be a b-multiplicative metric if the following axioms are satisfied:

- (μ1) $\mu(\alpha, \beta) > 1$ for all $\alpha, \beta \in A$ and $\mu(\alpha, \beta) = 1$ if and only if $\alpha = \beta$;
- (μ2) $\mu(\alpha, \beta) = \mu(\beta, \alpha)$ for all $\alpha, \beta \in A$;
- (μ3) $\mu(\alpha, z) \leq \mu(\alpha, \beta)^s \cdot \mu(\beta, \eta)^s$ for all $\alpha, \beta, \eta \in A$.

Then (A, μ, s) is said to be a b-multiplicative metric space.

➢ *Definition 2.2 [21]*

Given a set H having at least one element and $t \geq 1$. The function $\mu: H \times H \rightarrow [0, \infty)$ is called a strong b–metric space if

- $\mu(\alpha, \beta) = 0$ if and only if $\alpha = \beta$;
- $\mu(\alpha, \beta) = \mu(\alpha, \beta)$;
- $\mu(\alpha, \beta) \leq \mu(\alpha, z) + t\,\mu(z, \beta)$, for all $\alpha, \beta, z \in H$. The pair (H, μ) is called an (SbMS).

➢ *Definition 2.3 [21]*

Let H be a set that contains at least one element and $\zeta : H \times H \rightarrow [1, \infty)$. The function $\mu: H \times H \rightarrow [0, \infty)$ is called a (CSbMS) if

- $\mu(\alpha, \beta) = 0$ if and only if $\alpha = \beta$;
- $\mu(\alpha, \beta) = \mu(\alpha, \beta)$;
- $\mu(\alpha, \beta) \leq \mu(\alpha, z) + \zeta(\alpha, \beta)\,\mu(z, \beta)$, for all $\alpha, \beta, z \in H$.
- The pair (H, μ) is defined as a controlled strong b–metric space (CSbMS).
- Now we establish the idea of controlled strong b-multiplicative metric space (CSbMMS) as follows

➢ *Definition 2.4*

Let H be a set that contains at least one element and $\zeta : H \times H \rightarrow [1, \infty)$. The function $\mu: H \times H \rightarrow [1, \infty)$ is called a (CSbMMS) if

- $\mu(\alpha, \beta) = 1$ if and only if $\alpha = \beta$;

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- $\mu(\alpha, \beta) = \mu(\beta, \alpha)$;
- $\mu(\alpha, \beta) \leq \mu(\alpha, z) \,.\, \mu(z, \beta)^{\zeta(\alpha, \beta)}$, for all $\alpha, \beta, z \in H$.
- The pair (H, μ) is called a controlled strong b−multiplicative metric space (CSbMMS).
- Example 1. Let $H = [1, \infty)$ and define $\mu(\alpha, \beta) = 2^{\max\{|\alpha - \beta|, 2|\alpha - \beta| - 1\}}$ for every $\alpha, \beta \in H$ and $\zeta(\alpha, \beta) = \alpha + \beta + 2$.

➢ *Consider a sequence $\{\alpha_n\}$ in controlled strong b−metric space (CSbMMS) (H, μ).*

- $\{\alpha_n\}$ is called convergent to $\alpha \in H$, i.e. $\lim_n \alpha_n = \alpha$ if $\lim_n \mu(\alpha_n, \alpha)=1$.
- $\{\alpha_n\}$ is defined as a Cauchy sequence in H if $\lim_{n,m} \mu(\alpha_n, \alpha_m)=1$.
- CSbMMS (H, μ) is called complete if every Cauchy sequence in it is a convergent sequence.

➢ *Definition 4. Consider (H, μ) to be a (CSbMMS) by a functions. Take $y \in H$ along with $\xi > 0$.*

- An open ball $B(y, \eta)$ is $B(y, \eta) = \{x \in H, \mu(y, x) < \eta\}$.
- (ii) The mapping $S : H \to H$ is called continuous at $x \in H$ if $\forall\, \eta > 0, \exists\, \delta > 0$, satisfying $S(B(y, \delta)) \subseteq B(Sy, \eta)$.
- Obviously, if S is continuous at a point y in the (CSbMMS) (H, μ), then $y_n \to y$ implies that $Sy_n \to Sy$ as $n \to \infty$.

III. MAIN RESULTS

Presently, we are here ready to investigate a fixed point result parallel to the Banach contraction theorem in (CSbMMS).

➢ *Theorem Let (H, μ)) be a complete (CSbMMS) w.r.t. to the f mapping $\omega: H \times H \to [1, \infty)$. Suppose that $\Gamma: H \to H$ be a map satisfying $\mu(\Gamma\alpha, \Gamma\beta) \leq \mu(\alpha, \beta)^t$ for every $\alpha, \beta \in H$, where $t \in (0, 1)$.*

- For $\alpha_0 \in H$, lets have $\alpha_n = \Gamma^n\alpha_0$.
- Suppose that $\omega(\alpha_{i+1}, \alpha_m) < 1/t$.
- Also, for all $\alpha \in H$, $\lim_{n\to\infty} \omega(\alpha, \alpha_n)$ exists finitely.

➢ *Then the Mapping Γ Has a Unique Fixed Point.*

- Proof**:** In accordance with the hypothesis (1) of the theorem, take the sequence $\{\alpha_n = \Gamma^n \alpha_0\}$ in H. By the application of (1), we obtain $\mu(\alpha_n, \alpha_{n+1}) \leq \mu(\alpha_0, \alpha_1)^{t^n}$or every n in N.

➢ *For All Integers N, M with N < M, We Have*

$$\mu(\alpha_n, \alpha_m) \leq \mu(\alpha_n, \alpha_{n+1}).\, \mu(\alpha_{n+1}, \alpha_m)^{\omega(\alpha_{n+1}, \alpha_m)}$$

$$\leq \mu(\alpha_n, \alpha_{n+1}) \,.\, \mu(\alpha_{n+1}, \alpha_{n+2})^{\omega(\alpha_{n+1}, \alpha_m)} \,.\mu(\alpha_{n+2}, \alpha_m)^{\omega(\alpha_{n+1}, \alpha_m)\omega(\alpha_{n+2}, \alpha_m)}$$

$$\leq \mu(\alpha_n, \alpha_{n+1}) \,.\, \mu(\alpha_{n+1}, \alpha_{n+2})^{\omega(\alpha_{n+1}, \alpha_m)} \,.\, \mu(\alpha_{n+2}, \alpha_{n+3})^{\omega(\alpha_{n+1}, \alpha_m)\omega(\alpha_{n+2}, \alpha_m)} \,.\, \mu(\alpha_{n+2}, \alpha_{n+3})^{\omega(\alpha_{n+1}, \alpha_m)\omega(\alpha_{n+2}, \alpha_m)\omega(\alpha_{n+3}, \alpha_m)}$$

$$\leq \ldots\ldots\ldots\ldots\ldots$$

$$\leq \mu(\alpha_n, \alpha_{n+1}) \,.\, \prod_{i=n+1}^{m-2} \mu(\alpha_i, \alpha_{i+1})^{\prod_{j=n+1}^{i} \omega(\alpha_j, \alpha_m)} .\, \mu(\alpha_{m-1}, \alpha_m)^{\prod_{k=n+1}^{m-1} \omega(\alpha_k, \alpha_m)}$$

$$\leq \mu(\alpha_n, \alpha_{n+1}) \,.\, \prod_{i=n+1}^{m-2} \mu(\alpha_0, \alpha_1)^{t^i \prod_{j=n+1}^{i} \omega(\alpha_j, \alpha_m)} .\, \mu(\alpha_{m-1}, \alpha_m)^{\prod_{k=n+1}^{m-1} \omega(\alpha_k, \alpha_m)}$$

$$\leq \mu(\alpha_n, \alpha_{n+1}) \,.\, \prod_{i=n+1}^{m-1} \mu(\alpha_0, \alpha_1)^{t^i \prod_{j=n+1}^{i} \omega(\alpha_j, \alpha_m)}$$

$$\leq \mu(\alpha_n, \alpha_{n+1}) \,.\, \prod_{i=0}^{m-1} \mu(\alpha_0, \alpha_1)^{t^i \prod_{j=0}^{i} \omega(\alpha_j, \alpha_m)} = \mu(\alpha_n, \alpha_{n+1}).P_m \,.$$

Using ratio test for product with term

$$a_i = \mu(\alpha_0, \alpha_1)^{t^i \prod_{j=n+1}^{i} \omega(\alpha_j, \alpha_m)}$$

$$\frac{\log a_{i+1}}{\log a_i} = \frac{\log \mu(\alpha_0, \alpha_1)^{t^{i+1} \prod_{j=0}^{i+1} \omega(\alpha_j, \alpha_m)})}{\log \mu(\alpha_0, \alpha_1)^{t^{i} \prod_{j=0}^{i} \omega(\alpha_j, \alpha_m)})}$$

$$= \omega(\alpha_{i+1}, \alpha_m)\, t$$

$$< \frac{1}{t}\; t = 1.$$

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So that P_m is convergent and thus $\lim_{n\to\infty} P_m$ exists finitely. Also

$$\lim_{n\to\infty} \mu(\alpha_n, \alpha_{n+1}) \leq \lim_{n\to\infty} \mu(\alpha_0, \alpha_1)^{t^n} = 1.$$

Consequently $\lim_{n,m\to\infty} \mu(\alpha_n, \alpha_m) = 1$. Thus the sequence $\{\alpha_n\}$ is a Cauchy sequence.

Since (H, μ) is complete, there exists some α in H such that $\lim_{n\to\infty} \mu(\alpha_n, \alpha) = 1$ $\lim_{n\to\infty} \alpha_n = \alpha$.

Now we claim that Γα = α. For we have

$$\mu(\Gamma\alpha, \alpha) \leq \mu(\Gamma\alpha, \alpha_{n+1}).\ \mu(\alpha_{n+1}, \alpha)^{\omega(\alpha_{n+1},\ \alpha)}$$

$$= \mu(\Gamma\alpha, \Gamma\alpha_n).\ \mu(\alpha_{n+1}, \alpha)^{\omega(\alpha_{n+1},\ \alpha)}$$

$$\leq \mu(\alpha, \alpha_n)^t.\ \mu(\alpha_{n+1}, \alpha)^{\omega(\alpha_{n+1},\ \alpha)} \to 1 \text{ as } n \to \infty.$$

Which immediately implies that Γα = α. Suppose there is an element β in H such that Γβ = β and α ≠ β.

$$1 < \mu(\alpha, \beta) = \mu(\Gamma\alpha, \Gamma\beta) \leq \mu(\alpha, \beta)^t$$

which yields a contradiction. So α = β. Hence the theorem proved.

IV. CONCLUSION

The notion of controlled strong b-metric space introduced in this paper provide a platform for the study of fixed point theory in a new generalized metric space. The fixed point theorem proved here in the main result of this paper made the foundation of fixed point finding strategies on the introduced generalized metric space parallel to the flood of research after the Banach fixed point theorem.

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